Analysis of the Weibull Estimation for Competing Failure Modes

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SUMMARY & CONCLUSIONS

In this paper, the results of the parameter study using a simulation model for competing failure modes are discussed. For the investigation of the effect of one failure mode on the estimation of a second failure mode a model for competing failure modes has been developed. This model creates the possibility to define constant failure mode ratios between the two failure modes for the Monte Carlo simulation. Crucial part of this paper is the effect of the failure mode ratio \( r \) on the estimation of the shape parameter \( b \) and the characteristic lifetime \( T \). The results of these estimations are compared with the results of estimating failure mode A as a complete sample without consideration of censored data. Reason for this is to quantify the general effect on the estimation of taking competing failure modes as censored data into account. Further, the influence of the sample size in general is analyzed. Based on this parameter study and the results discussion, a recommendation for the analysis of systems with competing failure modes can be made.

1 INTRODUCTION

For durability and reliability verification of technical products, it is necessary to perform lifetime tests. A well-known issue of lifetime tests are competing failure modes. If competing failure modes occur in a system these failure modes have to be analyzed separately as described in [1], [2], [3] and [4]. Then separate lifetime distributions describe each failure mode. This is necessary because if different failure modes are not analyzed separately this may cause a kink in the Weibull probability plot, which is illustrated in Figure 1. This kink gets bigger with an increasing difference between two Weibull shape parameters \( b \) and will cause an imprecise estimation of the system reliability. The separate lifetime distributions are the input for the calculation of the system reliability. For the determination of a failure mode’s lifetime distribution, the failure times of the competing failure modes are usually treated as multiple censored data [5]. Based on the analysis of the different failure modes, the failure times of the system can be modeled using a series system. For this, it is necessary to prove that each failure mode is independent [4].

![Figure 1 - Weibull estimation for 2 failure modes](image-url)
In general competing failure modes can be modelled by generating a random failure time for each of the regarded failure modes. Then the failure times have to be compared for the definition of failure and survivor. Therefore, the first occurred failure mode is defined as failure and the second as suspension. This is quite a simple method for modelling competing failure modes but with one weakness for the investigation. This weakness is the ratio between failure modes, which is varying over 1000 samples. For the investigation, it is necessary to define a constant ratio between two failure modes.

The failure mode ratio $r$ is the results by the combination of the characteristic lifetimes and shape parameters for two failure modes. Because of this characteristic, the failure mode ratio $r$ is only constant for the total population of two failure modes. A topic in reliability engineering that deals with the same characteristic is the Stress-Strength-Interference (SSI). The SSI describes the relation between stress and strength for a product [1]. The overlapping area between stress and strength describes unreliability of a product as illustrated in Figure 2.

Equation (1) of the SSI consists of two integrals. The inner integral describes the function of strength. Transferred to reliability engineering this is equal with the survival probability $R(t)$. Thereby, in equation (1) the stress density function is multiplied with the survival probability $R(t)$ and integrated. The same principal now is used for the calculation of the shares $i_A$ and $i_B$ of failure mode A and B using the following equations (2) and (3).

$$i_A = \int_0^\infty f_A \cdot R_B \cdot dt$$  
$$i_B = \int_0^\infty f_B \cdot R_A \cdot dt$$  

Using equations (2) and (3) it is ensured that a certain failure mode only occurs if the other failure mode is survived. This fundamental relationship is also illustrated in Figure 3, which shows the relationship between two failure modes A and B. The shares of $i_A$ and $i_B$ for the system are the surfaces colored in yellow and red. By using this model for the total population of two failure modes a constant failure mode ration between the failure modes A and B can be defined, which is decisive to carry out general analysis.

For the modelling of competing failure modes, an analogy to the SSI is used. In the SSI, the reliability is the probability that the stress is smaller than the strength, if you compare a random sample out of the stress distribution and one of the strength distribution. The reliability may also be calculated analytically using equation (1) [1].

$$R = \int_{-\infty}^\infty f_B(\sigma) \cdot \left[\int_{\sigma}^\infty f_W(\sigma) \cdot d\sigma\right] \cdot d\sigma$$  

Transferred to competing failure modes each distribution would describe a failure mode. The reliability of the SSI is now the probability that the failure time of a random sample of failure mode A occurs earlier than a random sample of failure mode B. This ends in a system failure caused by failure mode A and describes the probability that failure mode A occurs earlier than failure mode B. The other way round, the unreliability in the SSI corresponds with the share of failures occurring because of failure mode B.

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2.2 Simulation Parameters

For investigating the effect of competing failure modes, several parameters may be investigated. As described before the focus of the investigation described in this paper is the failure mode ratio \( r \). Therefore, the restriction of investigating only two failure modes using the two parametric Weibull distributions was made. For both failure modes, the characteristic lifetime has to be defined. While \( T_A \) is defined constant as 1, \( T_B \) is one input parameter for the simulation and is varied from 0.25 to 5. Two additional input parameters for the simulation are the shape parameters \( b_A \) and \( b_B \) for both failure modes and are varied from 1 to 5. The failure mode ratio \( r \) is a result of defining the characteristic lifetime and the shape parameter for the failure modes A and B. For the investigation of competing failure modes it has to be considered that, the two shape parameters don’t become equal while varying their distributions. This is important because if the two failure modes are equal they are not independent from each other [4]. Another important input parameter for every reliability test and so for this investigation is the sample size \( n \), which will be varied from 10 to 100. An overview about the parameters varied in a full factorial test plan in the parameter study is illustrated in Figure 4.

![Figure 4 – Simulation parameters](image)

### 2.3 Simulation Process

After describing the modelling of competing failure modes and the input parameters for the simulation in the subchapters before, the simulation process will be described in this subchapter. The whole simulation process is illustrated as top-down process in Figure 5. As illustrated in this figure the simulation process starts with the definition of the total sample size \( n \) and the two failure modes A and B. Result of defining the characteristic lifetimes and shape parameters for the two failure modes is the lifetime ratio \( r \) between failure mode A and B. The shares of failure modes A and B and so the failure mode ratio \( r \) are calculated using the equations (2) and (3). The sample size for failure mode A \( n_A \) and for failure mode B \( n_B \) can then be calculated by multiplication the shares \( i_A \) and \( i_B \) with the total sample size \( n \). The calculated sample sizes for the defined failure modes A and B are then used as input for the random sampling of failure times. As described in subchapter 2.1 the random failure times are sampled based on the equations (4) and (5) using the rejections sampling method with a uniform distribution. In this sampling process, of course the assignment of failure mode A and B to each failure time is made. The Monte Carlo sampling for both failure modes is repeated 1000 cycles in the parameter study.

![Figure 5 – Simulation Process](image)
3 PARAMETER STUDY

In this chapter, the results of the parameter study Weibull estimation for competing failure modes will be described. Investigation parameters in this parameter study are \( b_A, b_B, T_B \) and \( n \). In Figure 4, the different levels for these investigation parameters are illustrated. Goal of this parameter study was to investigate the effect of the censoring share, as result of a failure mode \( B \), on the Weibull estimation of failure mode \( A \). Therefore, based on an example for competing failure modes the effect on the estimation of the Weibull shape parameter and the characteristic lifetime will be described in this chapter. Reason for this is the limited space in this paper, but the general results are discussed too. Additionally the effect of the sample size on the estimation will be described and a comparison between considering suspensions or not is made.

The estimators \( b_A \) and \( T_A \) are the results of the Weibull estimation for not considering suspensions and the estimators \( b_C \) and \( T_C \) are the results for considering suspensions. Another target figure of this investigation is the average deviation of \( b \) and \( T \). The censoring share is an input for this investigation and not a target figure.

In the example of the subchapters 3.1, 3.2 and 3.3 failure mode \( A \) has a shape parameter of \( b_A=3 \) and failure mode \( B \) of \( b_B=5 \). For the investigation of different censoring shares, \( T_B \) is varied between 0.25 and 5 while \( T_A=1 \) is constant. For the investigation of the effect on the estimation of \( b \) and \( T \) a total sample size of \( n=100 \) is chosen. A sample size of 100 was chosen because for small sample sizes and high censoring shares only a small number of data are usable and the censoring share can’t be varied as detailed as for \( n=100 \). Nevertheless, for the investigation of the effect of the sample size on the Weibull estimation, the total sample size is varied on the levels 100, 30 and 10 in subchapter 3.3.

3.1 Effect on the Estimation of \( b \)

The estimation of the correct shape parameter is decisive for the correct description of the failure behavior. Therefore, the estimation of the shape parameter for different failure mode ratios is estimated using multiple censored data as suspension or not. The failure mode ratio \( r \) in this case is described using the censoring share in percentage, which is the result of the combination of the default failure modes.

In Figure 6 the median-values for the estimators \( b_A \) and \( b_C \) are illustrated over the parameter \( T_B \) and over the censoring share. Additionally to the median-values the 90% quantile of all estimations for \( b_A \) and \( b_C \) are marked with dotted lines. The share of censoring dependent on \( T_B \) in this example is in maximum 99%. For a censoring share until 80%, the median-value of \( b_C \) is quite close to the default value of 3 and even the 90% quantile is quite narrow until a censoring share of 70%. For censoring shares above 80% \( b_C, \text{median} \) is getting worse and the 90% quantile is getting bigger especially for censoring shares above 90%. Estimation results of \( b_A, \text{median} \) are quite similar to \( b_C, \text{median} \) until a censoring share of 40% is reached. Above 40% of censoring the estimation of \( b_A, \text{median} \) is worse than \( b_C, \text{median} \).

3.2 Effect on the Estimation of \( T \)

The analysis of the effect on the estimation of the characteristic lifetime is done in the same way for the shape parameter \( b \). Again the median values, this time for the characteristic lifetime, \( T_A, \text{median} \) and \( T_C, \text{median} \), the 90% quantiles of the estimators and the average deviation referred to the default \( T_A \) are determined. \( T_A \) was defined for the investigation with \( T_A=1 \) and \( T_B \) again was varied to set different censoring shares. The results of the simulation for \( b_A=3, b_B=5 \) and \( n=100 \) is illustrated in the graph in figure 8. In this graph \( T_A, \text{median} \), \( T_C, \text{median} \), their 90% quantiles and the
censoring share are plotted over $T_B$. Also for high shares of censoring $T_{C, \text{median}}$ is very close to the default value of 1. Only for a censoring share over 90% small deviations can be observed. The 90% quantile of $T_{C, \text{median}}$ is until a censoring share of 70% smaller than 0.3. For censoring shares bigger than 70%, the 90% quantile arises quickly to a range of more than 1. $T_{A, \text{median}}$ is only for a censoring share of less than 10% close to the default value. For higher censoring shares, $T_A$ is getting smaller until at a share of 98% a $T_{A, \text{median}}$ of 0.2 which is a big difference compared to the default value. Interesting is that the 90% quantile of $T_A$ is over all variations of $T_B$ and the censoring time quite constant. This behavior for the estimation of the characteristic lifetime could be explained by the fact that the censoring times of failure mode occur at the end of the probability distribution of the failure mode A. If censoring times now are not being considered in the estimation, the characteristic lifetime will be estimated smaller than it truly is.

The average deviation for $T_A$ and $T_C$ referred to the default $T_A$ is illustrated in Figure 9 over the uncensored sample size equal to $n_A$. As expected, because of the analysis of $T_{A, \text{median}}$ and $T_{C, \text{median}}$, the average deviation for $T_{C, \text{median}}$ is much lower than for $T_{A, \text{median}}$ and for all sample sizes illustrated in the graph of Figure 9. While the average deviation for $T_C$ for a uncensored sample size of 10 is already smaller than 0.2, it is more than 0.6 for the average deviation of $T_A$.

Figure 8 – Estimation results for $T$

![Figure 8 – Estimation results for $T$](image)

Result for the parameter study for the characteristic lifetime is quite clear. If suspensions are considered in the estimation, than the estimation of $T$ is at least for a censoring share of 80% quite good. If suspensions are not considered in the estimation than estimations for $T$ with censoring shares over 10% are getting very bad, as can be seen in the figures 8 and 9.

3.3 Influence of the sample size

Last analysis for discussion in this paper is the analysis of the sample size influence. As mentioned in the subchapters 3.1 and 3.2, it is difficult to simulate many different censoring shares. Although, a large sample size is necessary to achieve good results using the maximum likelihood method. That is the reason for discussing the effect of the sample size separately in this subchapter. In Figure 10, the same example as before is plotted twice. The Graph on top shows the results for a total sample size of 30 and the graph below for 10. Figure 6 in subchapter 3.1 shows the results for a total sample size of 100. As expected, the 90% quantile is getting bigger when the sample size is getting smaller. The same effect can be observed for the quantiles of the characteristic lifetime. Another effect whether for the estimation of $b_A$, $b_B$, $T_A$ or $T_C$ cannot be observed in the parameter study.

![Figure 10 – Effect of total sample size](image)

4 SUMMARY

In this paper, a simulation process for the investigation of the Weibull estimation for competing failure is described. Crucial part of the simulation process is the modelling of competing failure modes. For the modelling of competing failure modes, a methodology was developed based on the approach of Stress-Strength-Interference. This approach was
successfully implemented in the simulation process and used to perform a parameter study. This study investigated the effects of competing failure modes on the Weibull estimation.

Based on the parameter study, the following recommendations for the analysis of competing failure modes can be made. It can be confirmed that for a successful analysis of competing failure modes the consideration of multiple censored data is necessary. The results in the parameter study considering suspensions are more precise than analyzing one failure mode as complete sample. This is decisive, especially for the estimation of the characteristic lifetime where otherwise big deviations from the default value may occur. A more amazing result of this parameter study is that, as long as suspension times are considered, the estimation results for the Weibull distribution are quite good as long as the censoring share is smaller than 80%. A high amount, which was not expected in the run-up to this study. The total sample size has no particular effect on the estimation of competing failure modes. However, as for standard reliability analysis, of course with a smaller sample size the 90% quantiles are getting bigger.

REFERENCES

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